Courant Institute of Mathematical Sciences

Magneto-Fluid Dynamics Division

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E. M. Barston

New York University

NYO- 1480-103
c.l.

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Abstract

Necessary and sufficient conditions for the
exponential stability of equilibria are obtained for
the resistive, viscous, incompressible MHD sheet pinch
in a gravitational field. Insulating as well as conduct-
ing boundaries are considered. The unperturbed magnetic
field $\vec{\overline{B}}_0$ (horizontal), resistivity η_0, viscosity ν_0,
and mass density ρ_0 are permitted to be arbitrary
(nonconstant) functions of the vertical coordinate
consistent with the equilibrium equations. In particular,
it is shown that if η_0 is not constant and the boundaries
are insulating, the pinch cannot be gravitationally
stabilized.

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Abstract

1.   Introduction.

Resistive plasma instabilities have received considerable study in recent years. In particular, instabilities in the resistive incompressible MHD sheet pinch have been investigated by several authors.[1-7] Most of this work has been restricted to the limiting cases of large or small resistivity.[1-3,6] Roberts and Taylor[4] considered purely gravitational resistive instabilities in a system with small constant shear and constant resistivity. Wei[5] has obtained sufficient conditions for stability in the special case of constant resistivity and viscosity. In this paper we investigate the exponential stability of static equilibria. Necessary and sufficient conditions for stability are obtained without any of the above mentioned approximations or limitations.

We adopt the model of reference 1, with the addition of an arbitrary nonconstant viscosity. The unperturbed state is assumed to be a static equilibrium solution of the model equations. The unperturbed magnetic field $\vec{B}_0$ (horizontal), resistivity η_0, viscosity ν_0, and mass density ρ_0 are permitted to be arbitrary functions of the vertical coordinate z consistent with the equilibrium equations.

Stability is discussed in terms of the L_2-norms of the system variables. The variable $\xi(\vec{x},t)$ is said to be exponentially stable if and only if for every $\varepsilon > 0$

there exists a constant M_ε such that $[\int |\xi|^2 dv]^{1/2} = \|\xi\|$ $\leq M_\varepsilon e^{\varepsilon t}$ for all $t \geq 0$. Necessary and sufficient conditions for exponential stability of the sheet pinch, as well as the maximal growth rate of the unstable pinch, are obtained directly from the theorems of reference 8 and Section 4 of this paper. These theorems extend the well-known energy principle for conservative systems to dissipative systems of a rather general class, and are valid for systems with discrete or continuous spectra. The maximal growth rate of an unstable system has been shown to be the supremum of a certain functional over the class of negative potential energy states.[8] Calculations of maximal growth rates for the unstable pinch will be presented in a subsequent paper.

In Section 2 we state the nonlinear model equations, derive the equilibria and the perturbation equations, and discuss boundary conditions. Section 3 begins with Theorem 3.1, which gives the necessary and sufficient conditions for the exponential stability of the sheet pinch, and we then proceed to deduce the consequences. Section 4 provides the basis for Theorem 3.1. Section 3 leads, in particular, to the following conclusions (the equilibrium quantities are scaled so that $\vec{B}_0(z)$ is fixed while $\hat{\eta}$, the magnitude of the resistivity, is varied):

1. The stability criteria (but not growth rates) are independent of $\nu_0(z)$ and $\hat{\eta}$.

2. If $g \frac{d\rho_0}{dz}$ is positive anywhere (g is the gravitational acceleration), the pinch is exponentially unstable. This holds for perfectly conducting or insulating boundaries.

3. If $\vec{B}_0^>$ or η_0 is constant, stability is completely equivalent to gravitational stability, i.e., the pinch is exponentially stable if and only if $g \frac{d\rho_0}{dz} \leq 0$. This holds for perfectly conducting or insulating boundaries.

4. For nonconstant η_0, the system is always unstable for sufficiently small wavenumbers with insulating boundaries, and cannot be gravitationally stabilized, but it can be gravitationally stabilized with perfectly conducting boundaries. Strictly speaking, however, perfectly conducting boundaries imply zero shear in the equilibrium magnetic field -- see Section 2(D).

5. Magnetic stabilization is possible with perfectly conducting boundary conditions. For a certain class of nonconstant η_0 (e.g., monotone) and no gravity, the imposition of a properly chosen constant zero-order magnetic field will stabilize a previously unstable system.

2. The Perturbation Equations

A. The Model

We consider an infinite horizontal layer of a magnetohydrodynamic fluid satisfying the following system of equations:

$$\nabla \cdot \vec{V} = 0 \tag{2.1}$$

$$\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho = 0 \tag{2.2}$$

$$\rho \left\{ \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} \right\} = - \vec{\nabla P} - \rho g \, \vec{e}_z + \vec{J} \times \vec{B} \tag{2.3}$$

$$+ \nabla^2 (\nu \vec{V}) - \vec{V} \nabla^2 \nu - (\nabla \times \vec{V}) \times \vec{\nabla} \nu$$

$$\vec{E} + \vec{V} \times \vec{B} = \eta \, \vec{J} \tag{2.4}$$

$$\frac{\partial \vec{B}}{\partial t} = - \nabla \times \vec{E} \tag{2.5}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} \tag{2.6}$$

$$\nabla \cdot \vec{B} = 0 \tag{2.7}$$

$$\frac{\partial \eta}{\partial t} + \vec{V} \cdot \vec{\nabla \eta} = 0 \tag{2.8}$$

$$\frac{\partial \nu}{\partial t} = ? \tag{2.9}$$

The equations are written in MKS units; μ_0 is the permeability of free space. The quantity $\rho(\vec{x}, t)$ denotes the mass density, $\vec{V}(\vec{x}, t)$ the fluid velocity, $\vec{B}(\vec{x}, t)$ the magnetic field, $P(\vec{x}, t)$ the scalar pressure,

$\vec{J}(\vec{x},t)$ the electric current density, $\nu(\vec{x},t)$ the viscosity, $\eta(\vec{x},t)$ the resistivity, g the gravitational acceleration, and $\vec{E}(\vec{x},t)$ the electric field. We choose a right-handed Cartesian coordinate system (x,y,z); z is the vertical coordinate with corresponding unit vector $\vec{e}_z$. The gravitational force is $-\rho g\vec{e}_z$. Provided the r.h.s. of Eq. (2.9) vanishes for the zero-order solutions of Section B, its precise form is of no importance in the sequel (see Sections B and C).

The quantities $\vec{E}$ and $\vec{J}$ may be eliminated in favor of $\vec{B}$ and $\vec{V}$ by means of Eqs. (2.4) and (2.6), resulting in the set of equations consisting of Eqs. (2.1), (2.2), (2.7), (2.8), (2.9) and

$$\rho\{\frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}\} = -\vec{\nabla}P - \rho g\,\vec{e}_z + \mu_0^{-1}(\nabla\times\vec{B})\times\vec{B}$$
$$+ \nabla^2(\nu\vec{V}) - \vec{V}\nabla^2\nu - (\nabla\times\vec{V})\times\vec{\nabla}\nu \qquad (2.3')$$

$$\frac{\partial\vec{B}}{\partial t} = -\nabla\times(\eta\nabla\times\vec{B})\mu_0^{-1} + \nabla\times(\vec{V}\times\vec{B}). \qquad (2.10)$$

$\vec{E}$ and $\vec{J}$ are obtained from $\vec{V}$ and $\vec{B}$ by means of

$$\vec{J} = \mu_0^{-1}\nabla\times\vec{B} \qquad (2.6')$$

$$\vec{E} = \mu_0^{-1}\eta\nabla\times\vec{B} - \vec{V}\times\vec{B} \qquad (2.4')$$

With the exception of our inclusion of a nonconstant viscosity ν, our model is identical to that considered in Ref. 1; the reader desiring further commentary on these equations is referred thereto.

B.   The Equilibrium

The boundaries of the fluid are taken to be $z = 0$ and $z = a$. The equilibrium quantities (distinguished by a 0-subscript) are assumed to be independent of t and the horizontal coordinates x and y; we take

$$\vec{V}_0 \equiv 0 \tag{2.11}$$

(no mass flow) and assume

$$\vec{B}_0 = B_{0x}(z)\vec{e}_x + B_{0y}(z)\vec{e}_y \tag{2.12}$$

With these limitations, the equilibrium quantities are allowed to be arbitrary functions of the vertical coordinate z consistent with the equations of Section A.   Thus $\rho_0(z)$, $\eta_0(z)$, and $\nu_0(z)$ are permitted to be arbitrarily prescribed functions of z on $[0,a]$ such that $\rho_0 > 0$, $\eta_0 > 0$, and $\nu_0 \geq 0$, and it is easily seen that the equations of Section A will then all be satisfied if and only if the remaining equilibrium quantities $\vec{E}_0$, $\vec{J}_0$, $\vec{B}_0$, and P_0 satisfy the following equations:

$$\vec{E}_0 = E_{0x}\vec{e}_x + E_{0y}\vec{e}_y = \text{constant} \tag{2.13}$$

(E_{0x} and E_{0y} are arbitrarily prescribed constants)

$$\vec{J}_0(z) = \eta_0^{-1}(z)\vec{E}_0 \tag{2.14}$$

$$\vec{B}_0(z) = \vec{B}_0(0) + \mu_0\vec{E}_0 \times \vec{e}_z \int_0^z \eta_0^{-1}(u)\,du \tag{2.15}$$

where $\overline{B}_0^>(0)$ is an arbitrarily prescribed (constant) horizontal field, and P_0 is given by

$$P_0(z) = P_0(0) + \int_0^z [(\overrightarrow{J}_0^> \times \overline{B}_0^>) \cdot \overrightarrow{e}_z^> - g\rho_0] \, du \qquad (2.16)$$

where $P_0(0)$ is an arbitrary constant. This completes the description of our equilibrium state, which is referred to as the "standard case" in Ref. 1.

C.   The Linearized Equations

The linearization of Eqs. (2.1), (2.2), (2.3'), (2.7), (2.8), (2.9), and (2.10) about the equilibrium solution of Section B leads respectively to the following equations for the perturbed quantities (denoted by subscript 1):

$$\nabla \cdot \overrightarrow{V}_1^> = 0 \qquad (2.17)$$

$$\frac{\partial \rho_1}{\partial t} + V_{1z}\rho_0'(z) = 0 \qquad (' \equiv \frac{\partial}{\partial z}) \qquad (2.18)$$

$$\rho_0 \frac{\partial \overline{V}_1^>}{\partial t} = - \overrightarrow{\nabla P}_1 - \rho_1 g\overrightarrow{e}_z^> + \mu_0^{-1}(\nabla \times \overline{B}_1^>) \times \overline{B}_0^> + \mu_0^{-1}(\nabla \times \overline{B}_0^>) \times \overline{B}_1^>$$
$$+ \nabla^2(\nu_0 \overrightarrow{V}_1^>) - \overrightarrow{V}_1^> \nu_0'' - (\nabla \times \overrightarrow{V}_1^>) \times \nu_0' \overrightarrow{e}_z^> \qquad (2.19)$$

$$\nabla \cdot \overline{B}_1^> = 0 \qquad (2.20)$$

$$\frac{\partial \eta_1}{\partial t} + V_{1z}\eta_0' = 0 \qquad (2.21)$$

$$\frac{\partial \nu_1}{\partial t} = ? \qquad (2.22)$$

and

$$\frac{\partial \vec{B}_1}{\partial t} = - \nabla \times (\eta_0 \nabla \times \vec{B}_1) \mu_0^{-1} - \mu_0^{-1} \nabla \times (\eta_1 \nabla \times \vec{B}_0) + \nabla \times (\vec{V}_1 \times \vec{B}_0). \quad (2.23)$$

Note that v_1 does not appear in Eqs. (2.17)-(2.21) since $\vec{V}_0 \equiv 0$.

The perturbed quantities are Fourier analyzed in the horizontal plane, i.e., we write each perturbed quantity $\phi_1(\vec{x}, t)$ in the form

$$\phi_1(\vec{x}, t) = \phi(z, t) e^{i(k_x x + k_y y)}. \quad (2.24)$$

Note the change in notation: A variable $(\rho, \vec{V}, \eta, P, \vec{J}, \vec{B})$ without a numerical subscript shall henceforth denote the transform (a function of z and t only) of the associated perturbed variable $(\rho_1, \vec{V}_1, \eta_1, P_1, \vec{J}_1, \vec{B}_1$ respectively).

The z-component of the curl of Eq. (2.19) gives

$$\rho_0 \dot{R} + L_1 R - i\mu_0 F J_z = q B_z \quad (2.25)$$

where $\dot{R} \equiv \frac{\partial R}{\partial t}$, $R' \equiv \frac{\partial R}{\partial z}$,

$$R(z, t) \equiv (\nabla \times \vec{V}_1)_z e^{-i(k_x x + k_y y)} = ik_x V_y(z, t) - ik_y V_x(z, t) \quad (2.26)$$

$$J_z(z, t) = \mu_0^{-1} (\nabla \times \vec{B}_1)_z e^{-i(k_x x + k_y y)} = \mu_0^{-1} (ik_x B_y - ik_y B_x) \quad (2.27)$$

$$F(z) \equiv \mu_0^{-1} [k_x B_{0x}(z) + k_y B_{0y}(z)] = \mu_0^{-1} \vec{k} \cdot \vec{B}_0 \quad (2.28)$$

$$q(z) \equiv i \mu_0^{-1} (k_x B_{0y}' - k_y B_{0x}') \quad (2.29)$$

$$L_1 \equiv - D\nu_0 D + k^2 \nu_0, \quad D \equiv \frac{d}{dz}, \quad k^2 \equiv k_x^2 + k_y^2. \quad (2.30)$$

We define the z-displacement $\xi(z, t)$ by

$$\xi(z,t) \equiv \int_0^t V_z(z,u) \, du + \xi_0(z) \qquad (2.31)$$

where $\xi_0(z)$ is an arbitrary function of z. The integration of Eqs. (2.18) and (2.21) with respect to t leads to

$$\rho(z,t) = \rho(z,0) + \rho_0' \xi_0 - \rho_0' \xi \qquad (2.32)$$

$$\eta(z,t) = \eta(z,0) + \eta_0' \xi_0 - \eta_0' \xi \qquad (2.33)$$

The z-component of the curl of Eq. (2.23) gives

$$\mu_0 \dot{J}_z + L_2 J_z - i\mu_0 FR = -q\mu_0 \dot{\xi} - \frac{d}{dz} \{q[\eta_0' \xi - \eta(z,0) - \eta_0' \xi_0]\} \qquad (2.34)$$

where

$$L_2 \equiv -D\eta_0 D + k^2 \eta_0 . \qquad (2.35)$$

The definition of R and Eq. (2.17) yields

$$V_x(z,t) = k^{-2}\{ik_x \dot{\xi}' + ik_y R\} \qquad (2.36)$$

$$V_y(z,t) = k^{-2}\{ik_y \dot{\xi}' - ik_x R\} \qquad (2.37)$$

and Eqs. (2.27) and (2.20) give

$$B_x(z,t) = k^{-2}\{ik_x B_z' + ik_y \mu_0 J_z\} \qquad (2.38)$$

$$B_y(z,t) = k^{-2}\{ik_y B_z' - ik_x \mu_0 J_z\} \qquad (2.39)$$

Given the two variables $\xi(z,t)$ and $B_z(z,t)$, J_z and R are determined by Eqs. (2.25) and (2.34), so that

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V_x, V_y, B_x and B_y can then be obtained from Eqs. (2.36)-(2.39).

We now proceed to obtain the equations determining the quantities $\xi(z,t)$ and $B_z(z,t)$. We multiply the x-component of Eq. (2.19) by ik_x and the y-component of Eq. (2.19) by ik_y, add the resulting equations, and by means of Eqs. (2.17) and (2.20) obtain

$$k^2 P = - \rho_0 \ddot{\xi}' + (\nu_0 \dot{\xi}')'' - k^2 \nu_0 \dot{\xi}' - \nu_0'' \dot{\xi}' + k^2 \nu_0' \dot{\xi} - \nu_0' \dot{\xi}''$$
$$- iF'B_z + iFB_z' - k^2 \mu_0^{-1} \vec{B} \cdot \vec{B}_0 \tag{2.40}$$

Substituting this expression for P into the z-component of Eq. (2.19) and using Eq. (2.32) we find

$$L_3 \ddot{\xi} + L_4 \dot{\xi} - gk^2 \rho_0' \xi + iF[B_z'' - k^2 B_z] - iF''B_z$$
$$= - gk^2 [\rho(z,0) + \rho_0' \xi_0] \tag{2.41}$$

where

$$L_3 \equiv - D\rho_0 D + k^2 \rho_0 \tag{2.42}$$

$$L_4 \equiv D^2 \nu_0 D^2 - 2k^2 D\nu_0 D + k^4 \nu_0 + k^2 \nu_0'' \tag{2.43}$$

The z-component of Eq. (2.23) and Eqs. (2.17) and (2.33) imply

$$\eta_0^{-1} \dot{B}_z + L_5 B_z - i\mu_0 F \eta_0^{-1} \dot{\xi} - iF' \eta_0' \eta_0^{-1} \xi$$
$$= - i F' \eta_0^{-1} [\eta(z,0) + \eta_0' \xi_0] \tag{2.44}$$

where

$$L_5 = \mu_0^{-1}[-D^2 + k^2]$$
(2.45)

The term $iF(B_z'' - k^2 B_z) = -iF\mu_0 L_5 B_z$ is eliminated from Eq. (2.41) by means of Eq. (2.44), and we write the resulting equation together with Eq. (2.44) as the single matrix equation

$$
\begin{pmatrix} L_3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{\xi} \\ \ddot{B}_z \end{pmatrix} + \begin{pmatrix} L_4 + \dfrac{\mu_0^2 F^2}{\eta_0} & \dfrac{\mu_0 iF}{\eta_0} \\ -\dfrac{\mu_0 iF}{\eta_0} & \eta_0^{-1} \end{pmatrix} \begin{pmatrix} \dot{\xi} \\ \dot{B}_z \end{pmatrix} +
$$

$$
+ \begin{pmatrix} \dfrac{\mu_0 FF'\eta_0'}{\eta_0} - gk^2 \rho_0' & -iF'' \\ -\dfrac{iF'\eta_0'}{\eta_0} & L_5 \end{pmatrix} \begin{pmatrix} \xi \\ B_z \end{pmatrix}
$$
(2.46)

$$
= \begin{pmatrix} -gk^2[\rho(z,0)+\rho_0'\xi_0]+\mu_0\eta_0^{-1}FF'[\eta(z,0)+\eta_0'\xi_0] \\ -iF'\eta_0^{-1}[\eta(z,0)+\eta_0'\xi_0] \end{pmatrix}
$$

Eqs. (2.13) (2.15), and (2.28) imply

$$F'\eta_0 = \mu_0^{-1}[k_x B_{0x}'\eta_0 + k_y B_{0y}'\eta_0] = k_x E_{0y} - k_y E_{0x} = \text{const.}$$
(2.47)

so that

$$0 = (F'\eta_0)' = F''\eta_0 + F'\eta_0' \iff F'' = -\frac{F'\eta_0'}{\eta_0}$$
(2.48)

This completes the derivation of the equations for the perturbed quantities. The solution of Eqs. (2.17)-(2.23) is accomplished by solving Eq. (2.46) for ξ and B_z;

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the remaining variables are then obtained from Eqs. (2.25), (2.32)-(2.34), and Eqs. (2.36)-(2.40).

D. Boundary Conditions

The boundary conditions commonly employed in the literature are those associated with rigid perfectly conducting walls. These lead to the simplest boundary conditions on B_z, namely that $B_z(0,t) = B_z(a,t) = 0$. However, perfect conductors require that $\vec{E}_{tan} = 0$ thereon, so that Eqs. (2.13) and (2.15) would require $\vec{E}_0 \equiv 0$ and $\vec{B}_0(z) = \vec{B}_0(0) = $ constant, i.e., the assumption of perfectly conducting boundaries implies zero shear in the equilibrium magnetic field $\vec{B}_0(z)$. To avoid this specialization we shall assume <u>rigid perfectly insulating</u> walls at $z = 0$ and $z = a$.[*] Thus we require that

$$J_z(0,t) = J_z(a,t) = 0 \tag{2.49}$$

$$V_z(0,t) = V_z(a,t) = 0 \tag{2.50}$$

For $\nu_0 > 0$, we also require that

$$V_x(0,t) = V_y(0,t) = V_x(a,t) = V_y(a,t) = 0 \tag{2.51}$$

which together with Eq. (2.17) implies

$$V_z'(0,t) = V_z'(a,t) = 0 \tag{2.52}$$

---

[*] Perfectly conducting boundary conditions, imposed on the perturbed quantities only (i.e., $\vec{E}_0 \neq 0$), will nevertheless formally be considered in Section 3, due to their common usage in the literature.

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Eqs. (2.26) and (2.51) lead to

$$R(0,t) = R(a,t) = 0 \tag{2.53}$$

while Eqs. (2.31), (2.50), and (2.52) give

$$\dot{\xi}(0,t) = \dot{\xi}(a,t) = 0 \tag{2.54}$$

and

$$\dot{\xi}'(0,t) = \dot{\xi}'(a,t) = 0 \tag{2.55}$$

We shall require that the arbitrary function $\xi_0(z)$ appearing in Eq. (2.31) satisfy

$$\xi_0(0) = \xi_0(a) = 0 \tag{2.56}$$

so that Eqs. (2.31) and (2.50) yield

$$\xi(0,t) = \xi(a,t) = 0 \; . \tag{2.57}$$

It remains to obtain the boundary conditions on $B_z(z,t)$. We assume that Eqs. (2.6) and (2.7) hold everywhere and that there are no surface currents at the walls. These conditions then imply that B_z and $\vec{B}_{tan}$ are continuous at the walls, i.e.,

$$B_x(0+,t) = B_x(0-,t), \quad B_y(0+,t) = B_y(0-,t)$$
$$B_x(a+,t) = B_x(a-,t), \quad B_y(a+,t) = B_y(a-,t) \tag{2.58}$$

and

$$B_z(0+,t) = B_z(0-,t), \quad B_z(a+,t) = B_z(a-,t) \tag{2.59}$$

Eqs. (2.20) and (2.58) lead to

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$$B_z'(0+,t) = B_z'(0-,t), \qquad B_z'(a+,t) = B_z'(a-,t) \qquad (2.60)$$

Thus both B_z and B_z' are continuous at $z = 0$ and $z = a$. Outside the fluid, the equations $\nabla \cdot \vec{B}_1 = 0$ and $\nabla \times \vec{B}_1 = \mu_0 \vec{J}_1 = 0$ imply that

$$\vec{B}_1 = \vec{\nabla} \phi_1 , \qquad \nabla^2 \phi_1 = 0 , \qquad z < 0, \; z > a \qquad (2.61)$$

and since we have Fourier transformed in the horizontal plane, we set $\phi_1(\vec{x},t) = \phi(z,t)e^{i(k_x x + k_y y)}$ and obtain from the last of Eqs. (2.61)

$$\phi''(z,t) - k^2 \phi(z,t) = 0 , \qquad z < 0, \; z > a \qquad (2.62)$$

with solution $\phi(z,t) = h(t)e^{\pm kz}$. Therefore

$$B_z(z,t) = \begin{cases} -kh_1(t)e^{-kz} , & z > a \\[2ex] kh_2(t)e^{kz} , & z < 0 \end{cases} \qquad (2.63)$$

where $k \equiv \sqrt{k_x^2 + k_y^2} = \sqrt{k^2}$, so that

$$B_z'(a+,t) = -kB_z(a+,t), \quad B_z'(0-,t) = kB_z(0-,t) \qquad (2.64)$$

From the continuity of B_z and B_z' at 0 and a we obtain the boundary conditions

$$B_z'(a-,t) + kB_z(a-,t) = 0, \qquad B_z'(0+,t) - kB_z(0+,t) = 0 \quad (2.65)$$

The magnetic field $B_z(z,t)$ outside the fluid is given by

$$B_z(z,t) = \begin{cases} B_z(a,t)e^{-k(z-a)}, & z \geq a \\[2ex] B_z(0,t)e^{kz} , & z \leq 0 . \end{cases} \qquad (2.66)$$

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We have completed the derivation of the necessary boundary conditions, and are now in a position to specify the domains D_n of the operators L_n ($n = 1,2,3,4,5$) defined previously. We assume henceforth that $\rho_0(z)$, $\eta_0(z)$, and $\nu_0(z)$ are all in $C^2[0,a]$ (the class of all twice continuously differentiable functions on $[0,a]$). In view of boundary conditions (2.53), (2.49), (2.54), (2.55) and (2.65) we take

$$D_1 \equiv D_2 \equiv D_3 \equiv \{f(z) \mid f \in C^2[0,a], \; f(0) = f(a) = 0\} \qquad (2.67)$$

$$D_4 \equiv \{f(z) \mid f \in C^4[0,a], \; f(0) = f(a) = f'(0) = f'(a) = 0\} \qquad (2.68)$$

$$D_5 \equiv \{f(z) \mid f \in C^2[0,a], \; f'(a) + kf(a) = 0 = f'(0) - kf(0)\} \qquad (2.69)$$

The operators L_n are all symmetric on D_n and in particular $L_1 \geq 0$, $L_2 > 0$, $L_3 > 0$, $L_5 \geq 0$ and we shall assume that $L_4 \geq 0$ (it suffices for example that $\nu_0'' \geq 0$ on $[0,a]$).

E. Dimensionless Variables

We proceed to write Eq. (2.46) in dimensionless form. To this end, we introduce the following definitions:

$$\zeta \equiv a^{-1}z , \qquad 0 \leq z \leq a \qquad (2.70)$$

$$\hat{\eta} \equiv \max_{[0,a]} \eta_0(z), \qquad \hat{\nu} \equiv \max_{[0,a]} \nu_0(z) , \qquad \hat{\rho} \equiv \max_{[0,a]} \rho_0(z) \qquad (2.71)$$

$$f(\zeta) \equiv \hat{\eta}^{-1}\eta_0(a\zeta), \qquad h(\zeta) \equiv \hat{\rho}^{-1}\rho_0(a\zeta), \qquad s(\zeta) \equiv \hat{\nu}^{-1}\nu_0(a\zeta) \qquad (2.72)$$

$$\vec{J}^{>*} \equiv \hat{\eta}^{-1}\vec{E}_0^{>} \quad \text{(constant)} \tag{2.73}$$

and assuming $J^* \equiv |\vec{J}^{>*}| > 0$,

$$\vec{b}^{>}(\zeta) \equiv \frac{\vec{B}_0^{>}(a\zeta)}{a\mu_0 J^*} = \vec{\beta}^{>} + \vec{e}_j^{>} \times \vec{e}_z^{>} \int_0^{\zeta} \frac{dr}{f(r)} \tag{2.74}$$

$$\vec{\beta}^{>} \equiv \frac{\vec{B}_0^{>}(0)}{a\mu_0 J^*}, \qquad \vec{e}_j^{>} \equiv \frac{\vec{J}^{>*}}{J^*} = \frac{\vec{E}_0^{>}}{|\vec{E}_0^{>}|} \tag{2.75}$$

$$\tilde{\eta} \equiv a^2 \mu_0^{3/2} J^* \hat{\rho}^{-1/2}, \qquad T \equiv (\hat{\rho}/\mu_0)^{1/2}/J^* \tag{2.76}$$

$$\vec{\kappa}^{>} = \vec{k}^{>}a, \quad \kappa \equiv ka, \quad \gamma \equiv \frac{g\hat{\rho}}{a\mu_0(J^*)^2}, \quad u \equiv \frac{t}{T}$$

$$W(\zeta, u) \equiv a^{-1}\xi(a\zeta, uT) \tag{2.77}$$

$$\psi(\zeta, u) \equiv (a\mu_0 J^*)^{-1} B_z(a\zeta, uT) . \tag{2.78}$$

These definitions are valid for $J^* \neq 0$ (nonzero magnetic shear) which represents the most interesting case. Indeed, $\vec{J}^{>*} = 0$ implies that $\vec{B}_0^{>}$ and F are constants, and it will be shown in the sequel that exponential stability of a zero shear system is equivalent to gravitational stability, i.e., the system is exponentially stable if and only if $\rho_0' \leq 0$ on $[0,a]$. Thus the equilibrium magnetic field $\vec{B}_0^{>}$ can affect exponential stability only if it possesses nonzero shear ($J^* \neq 0$).

Equations (2.14), (2.72), and (2.73) imply

$$\vec{J}_0^{>}(z) = f^{-1}(\frac{z}{a})\vec{J}^{>*} \tag{2.79}$$

Note that the equilibrium quantities have been scaled so that by keeping $\overline{J}^{>*}$ fixed while letting $\hat{\eta}$ vary, $\overline{J}_0^>$, $\overline{B}_0^>$, and $\overline{b}^>(\zeta)$ remain fixed independent of $\hat{\eta}$.

Equation (2.46) can now be expressed in terms of the dimensionless two-vector Θ, defined by

$$\Theta(\zeta,u) \equiv \begin{pmatrix} W(\zeta,u) \\ \psi(\zeta,u) \end{pmatrix} \qquad (2.80)$$

and takes the form

$$P\ddot{\Theta} + K\dot{\Theta} + H\Theta = C \qquad (2.81)$$

where $\dot{\Theta} \equiv \dfrac{\partial \Theta}{\partial u}$,

$$P = \begin{pmatrix} \tilde{L}_3 & 0 \\ 0 & 0 \end{pmatrix} \qquad (2.82)$$

$$K = \begin{pmatrix} \tilde{L}_4 & 0 \\ 0 & 0 \end{pmatrix} + \frac{\tilde{\eta}}{\hat{\eta}f} \begin{pmatrix} (\vec{\kappa}\cdot\vec{b})^2 & i\vec{\kappa}\cdot\vec{b} \\ -i\vec{\kappa}\cdot\vec{b} & 1 \end{pmatrix} \qquad (2.83)$$

$$H = \begin{pmatrix} G & i\kappa r \\ -i\kappa r & \tilde{L}_5 \end{pmatrix} \qquad (2.84)$$

(We have used equation (2.48) to put H in symmetric form.)

$$r = (\vec{e}_\kappa \times \vec{e}_j)\cdot\vec{e}_z f' f^{-2} \qquad (' \equiv \tfrac{\partial}{\partial\zeta}) \qquad (2.85)$$

$$G = \kappa^2[(\vec{e}_\kappa\cdot\vec{b}^>)r - \gamma h'(\zeta)] \qquad (2.86)$$

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$$\tilde{L}_3 = -\frac{d}{d\zeta} h \frac{d}{d\zeta} + \kappa^2 h, \quad \tilde{L}_5 = -\frac{d^2}{d\zeta^2} + \kappa^2 \tag{2.87}$$

$$\tilde{L}_4 = \delta\left[\frac{d^2}{d\zeta^2} s \frac{d^2}{d\zeta^2} - 2\kappa^2 \frac{d}{d\zeta} s \frac{d}{d\zeta} + \kappa^4 s + \kappa^2 s''\right] \tag{2.88}$$

$$\vec{e}_\kappa = \frac{\vec{\kappa}}{\kappa}, \quad \delta = \frac{\hat{\gamma}_T}{a^2 \hat{\rho}} = \frac{\hat{v}}{a^2 J^* (\hat{\rho}\mu_0)^{1/2}} \tag{2.89}$$

$$C = \begin{pmatrix} GW(\zeta,0)+\kappa^2\{(\vec{e}_\kappa\cdot\vec{b})(\vec{e}_\kappa\times\vec{e}_j)\cdot\vec{e}_z \eta(a\zeta,0)f^{-2}\hat{\eta}^{-1}-\gamma\rho(a\zeta,0)\hat{\rho}^{-1}\} \\ -i\kappa f^{-2}(\vec{e}_\kappa\times\vec{e}_j)\cdot\vec{e}_z[f'W(\zeta,0)+\hat{\eta}^{-1}\eta(a\zeta,0)] \end{pmatrix} \tag{2.90}$$

The domains $\tilde{D}_n$ of the operators $\tilde{L}_n$ $(n = 3,4,5)$ are, from definitions (2.67), (2.68), and (2.69),

$$\tilde{D}_3 = \{f(\zeta) \mid f \in C^2[0,1], \ f(0) = f(1) = 0\} \tag{2.91}$$

$$\tilde{D}_4 = \{f(\zeta) \mid f \in C^4[0,1], \ f(0)=f(1)=f'(0)=f'(1)=0\} \tag{2.92}$$

$$\tilde{D}_5 = \{f(\zeta) \mid f \in C^2[0,1], f'(1)+\kappa f(1)=0=f'(0)-\kappa f(0)\} \tag{2.93}$$

For each fixed u, $\Theta(\zeta,u)$ regarded as a function of ζ is an element of the Hilbert space $\mathcal{H} = L_2[0,1] \times L_2[0,1]$, and the domains D_P, D_K, and D_H of the linear operators P, K, and H are the following subsets of $\mathcal{H}$:

$$D_P = \tilde{D}_3 \times L_2[0,1] \tag{2.94}$$

$$D_K = \begin{cases} \tilde{D}_4 \times L_2[0,1], & \hat{v}s > 0 \text{ on } [0,1] \\ \mathcal{H}, & \hat{v} = 0 \end{cases} \tag{2.95}$$

$$D_H = L_2[0,1] \times \tilde{D}_5. \tag{2.96}$$

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The inner product $\langle , \rangle$ on $\mathcal{H}$ is of course given by

$$\langle \theta_1, \theta_2 \rangle = (W_1, W_2) + (\psi_1, \psi_2) \tag{2.97}$$

with $(W_1, W_2) = \int_0^1 \overline{W}_1 W_2 \, d\zeta,$

and it is easily seen that P, K, and H are symmetric on their domains of definition. Furthermore, we have

$$\langle \theta, P\theta \rangle = \int_0^1 h\{|W'|^2 + \kappa^2 |W|^2\} \, d\zeta \geq 0 \tag{2.98}$$

and

$$\langle \theta, K\theta \rangle = (W, \tilde{L}_4 W) + (\tilde{\eta}/\hat{\eta}) \int_0^1 f^{-1} |(\vec{\kappa} \cdot \vec{b})W + i\psi|^2 d\zeta \tag{2.99}$$

so that $\tilde{L}_4 \geq 0$ implies that $K \geq 0$.

3. Stability

It will be shown in Section 4 that the sheet pinch is exponentially stable to disturbances with wave number $\vec{k}^> = \vec{\kappa}^> a^{-1}$ if and only if $H = H(\vec{\kappa}^>) \geq 0$ for that value of $\vec{\kappa}^>$, so that the pinch is exponentially stable to arbitrary disturbances if and only if $H \geq 0$ for arbitrary $\vec{\kappa}^>$. Necessary and sufficient conditions that H be positive semidefinite will now be given, and several conclusions regarding stability will be drawn therefrom. Since H is independent of ν_0 and $\hat{\eta}$, the stability criteria (but not the growth rates) will of course be independent of ν_0 and $\hat{\eta}$ ($\hat{\eta} > 0$).

We define the functional $F_\varepsilon[\psi]$ for all real $\psi(\zeta) \in \tilde{D}_5$ and all $\varepsilon > 0$ such that $G_\varepsilon \equiv G + \varepsilon > 0$ on [0,1] by

$$F_\varepsilon[\psi] \equiv \int_0^1 \{(\psi')^2 + \kappa^2[1 - \frac{r^2}{G_\varepsilon}]\psi^2\} \, d\zeta + \kappa[\psi^2(0) + \psi^2(1)] \, . \qquad (3.1)$$

__Theorem__ 3.1. $H(\vec{\kappa}^>) \geq 0$ on D_H if and only if $G \geq 0$ on [0,1] and $F_\varepsilon[\psi] \geq 0$ for all real $\psi \in \tilde{D}_5$ and all $\varepsilon > 0$.

__Proof:__ Our previous assumption that $\rho_0 \in C^2[0,a]$ and $\eta_0 \in C^2[0,a]$ implies that $G = G(\vec{\kappa}^>, \zeta) \in C[0,1]$. The necessity that $G \geq 0$ on [0,1] is obvious. Suppose then that $G \geq 0$ on [0,1]. Let $W = W_1 + iW_2 \in L_2[0,1]$,

$\psi = \psi_1 + i\psi_2 \in \tilde{D}_5$, and $\Theta = \begin{pmatrix} W \\ \psi \end{pmatrix}$. Then for $\varepsilon > 0$,

$$\langle \Theta, H\Theta \rangle = \int_0^1 \{\overline{W}[GW + i\kappa r\psi] + \overline{\psi}[-i\kappa rW + \tilde{L}_5\psi]\}\, d\zeta$$

$$= \int_0^1 \{G[W_1^2 + W_2^2] + 2\kappa r[W_2\psi_1 - W_1\psi_2] + |\psi'|^2 + \kappa^2|\psi|^2\}d\zeta - \psi'\overline{\psi}\Big|_0^1$$

$$= \int_0^1 \{G_\varepsilon[W_1 - \frac{\kappa r\psi_2}{G_\varepsilon}]^2 - \varepsilon W_1^2\}\, d\zeta + F_\varepsilon[\psi_2] \qquad (3.2)$$

$$+ \int_0^1 \{G_\varepsilon[W_2 + \frac{\kappa r\psi_1}{G_\varepsilon}]^2 - \varepsilon W_2^2\}\, d\zeta + F_\varepsilon[\psi_1]$$

$$= R_-[\varepsilon, W_1, \psi_2] + F_\varepsilon[\psi_2] + R_+[\varepsilon, W_2, \psi_1] + F_\varepsilon[\psi_1]$$

where $\quad R_\pm[\varepsilon, W, \psi] \equiv \int_0^1 \{G_\varepsilon[W \pm \frac{\kappa r\psi}{G_\varepsilon}]^2 - \varepsilon W^2\}\, d\zeta$.

Now suppose there exists a real $\psi_2(\zeta) \in \tilde{D}_5$ such that $F_\varepsilon[\psi_2] = \Delta < 0$, for some $\varepsilon > 0$. Then there exists a real $W_1(\zeta) \in C^\infty[0,1]$ which vanishes identically outside some closed subinterval of $(0,1)$ such that $\int_0^1 G_\varepsilon[W_1 - \frac{\kappa r\psi_2}{G_\varepsilon}]^2\, d\zeta < \frac{|\Delta|}{2}$. Choosing $\psi_1 \equiv 0$ and $W_2 \equiv 0$, equation (3.2) leads to $\langle \Theta, H\Theta \rangle < \frac{\Delta}{2} < 0$. Thus $H \geq 0$ implies $G \geq 0$ on $[0,1]$ and $F_\varepsilon[\psi] \geq 0$ for all $\varepsilon > 0$ and all real $\psi \in \tilde{D}_5$.

Now suppose $\langle \Theta, H\Theta \rangle = \Delta < 0$ for some $\Theta = \begin{pmatrix} W_1 + iW_2 \\ \psi_1 + i\psi_2 \end{pmatrix}$ $\in L_2[0,1] \times \tilde{D}_5$. Then by equation (3.2),

$$\Delta = R_-[\varepsilon, W_1, \psi_2] + R_+[\varepsilon, W_2, \psi_1] + F_\varepsilon[\psi_1] + F_\varepsilon[\psi_2]$$

and this holds for all $\varepsilon > 0$. Clearly for some sufficiently small $\varepsilon > 0$ we have

$$R_-[\varepsilon, W_1, \psi_2] + R_+[\varepsilon, W_2, \psi_1] \geq \frac{\Delta}{2}$$

so that equation (3.2) yields

$$F_\varepsilon[\psi_1] + F_\varepsilon[\psi_2] \leq \frac{\Delta}{2} < 0 ,$$

which implies that $F_\varepsilon[\psi] < 0$ for $\psi = \psi_1$ or $\psi = \psi_2$. Hence $G \geq 0$ on $[0,1]$ and $F_\varepsilon[\psi] \geq 0$ for all real $\psi \in \tilde{D}_5$ and all $\varepsilon > 0$ implies that $H \geq 0$. This completes the proof.

We note that all the preceding statements of this section remain valid for the boundary conditions $\psi(0) = \psi(1) = 0$ corresponding to perfectly conducting boundaries, provided only that $\tilde{D}_5$ is replaced by $\tilde{D}_5' \equiv \{\psi \mid \psi \in C^2[0,1], \psi(0) = \psi(1) = 0\}$.

The quantity r, defined in equation (2.85), vanishes identically in ζ if $f(\zeta)$ is constant or if $\vec{\kappa}$ is parallel to $\vec{J}^*$. In this case,

$$H = \begin{pmatrix} -\kappa^2 \gamma h' & 0 \\ 0 & \tilde{L}_5 \end{pmatrix} \qquad (3.3)$$

and

$$\langle \Theta, H\Theta \rangle = -\kappa^2 \gamma (W, h'W) + (\psi, \tilde{L}_5 \psi) .$$

Since $\tilde{L}_5 \geq 0$ on $\tilde{D}_5$ and $\tilde{D}_5'$, $H \geq 0$ if and only if $h'(\zeta) \leq 0$ on $[0,1]$, for $\kappa^2 \gamma > 0$. Note that if $J^* = 0$ (i.e., $\overline{J}_0^{>} \equiv 0$), then $F'(z) \equiv 0$ and the operator corresponding to H in equation (2.46) reduces to

$$\begin{pmatrix} - gk^2 \rho_0' & 0 \\ 0 & L_5 \end{pmatrix}$$

and, as above, would be nonnegative definite if and only if $\rho_0'(z) \leq 0$ on $[0,a]$, for $k^2 g > 0$. Thus if either $\overline{J}_0^{>} \equiv 0$ or η_0 is constant, then exponential stability is equivalent to gravitational stability, i.e., the sheet pinch is exponentially stable (for all $\overline{k}^{>}$) if and only if $g\rho_0'(z) \leq 0$ on $[0,a]$, and this holds for perfectly conducting as well as insulating boundaries. If $g\rho_0'(z) > 0$ for some $z \in [0,a]$, the pinch will be exponentially unstable to disturbances with $\overline{k}^{>}$ parallel to $\overline{J}_0^{>}$ (regardless of whether $\overline{J}_0^{>} \equiv 0$ or η_0 is constant).

The preceding paragraph demonstrated the necessity that $g \rho_0'(z) \leq 0$ on $[0,a]$ for the sheet pinch to be stable (for arbitrary $\overline{k}^{>}$). In general, with the exception of the cases $\overline{J}_0^{>} \equiv 0$ (no shear) and constant η_0, it is not sufficient. To see this, we write

$$G = \kappa^2 [Q_\zeta(\overline{e}_\kappa^{>}) f' f^{-2} - \gamma h']$$
(3.4)

where

$$Q_\zeta(\overrightarrow{e}_\kappa) \equiv [\overrightarrow{e}_\kappa \cdot \overrightarrow{b}(\zeta)](\overrightarrow{e}_\kappa \times \overrightarrow{e}_j) \cdot \overrightarrow{e}_z \ , \quad 0 \leq \zeta \leq 1 \tag{3.5}$$

Let $p(\zeta) = \displaystyle\int_0^\zeta f^{-1}(r) \, dr$, so that by equation (2.74),

$$\overrightarrow{b}(\zeta) = \overline{\beta} + p(\zeta)\overrightarrow{e}_j \times \overrightarrow{e}_z \tag{3.6}$$

Inserting this into equation (3.5), we find

$$4Q_0(\overrightarrow{e}_\kappa) = (\overrightarrow{e}_\kappa \cdot [\overrightarrow{e}_j \times \overrightarrow{e}_z + \overline{\beta}])^2 - (\overrightarrow{e}_\kappa \cdot [\overrightarrow{e}_j \times \overrightarrow{e}_z - \overline{\beta}])^2 \tag{3.7}$$

and for $\zeta > 0$,

$$Q_\zeta(\overrightarrow{e}_\kappa) = p^{-1}\{(\overrightarrow{e}_\kappa \cdot [p\overrightarrow{e}_j \times \overrightarrow{e}_z + \frac{\overline{\beta}}{2}])^2 - (\overrightarrow{e}_\kappa \cdot \frac{\overline{\beta}}{2})^2\} \tag{3.8}$$

It is immediately apparent that if $\overline{\beta}$ is not parallel to $\overrightarrow{e}_j \times \overrightarrow{e}_z$, i.e., if $\overrightarrow{b}$ is nonunidirectional, then for each fixed $\zeta \in [0,1]$, $Q_\zeta(\overrightarrow{e}_\kappa)$ assumes both positive and negative values as $\overrightarrow{e}_\kappa$ varies. Thus even in the circumstance that $g\rho_0'(\zeta) \equiv 0$, $\overline{\beta} \cdot \overrightarrow{e}_j \neq 0$ and $f'(\hat{\zeta}) \neq 0$ for some $\hat{\zeta} \in [0,1]$ guarantee that $G < 0$ at the point $\hat{\zeta}$ for some range of values of $\overrightarrow{e}_\kappa$ — the system would then be unstable to disturbances possessing those values of $\overrightarrow{e}_\kappa$. In other words, nonconstant η_0, constant ρ_0 (or no gravity), and $\overrightarrow{B}_0(z)$ nonunidirectional imply exponential instability, and this holds for perfectly conducting or insulating boundaries. Indeed, if $\overrightarrow{B}_0(z)$ is nonunidirectional, the system can be made unstable for any $g\rho_0(z)$ by choosing an $\eta_0(z)$ with sufficiently large gradients. Suppose now that $\overline{\beta}$ is parallel to

$\vec{e}_j \times \vec{e}_z$ and that $g\rho_0' \equiv 0$. Let $\vec{e}_j = \vec{e}_y$, so that
$\vec{e}_j \times \vec{e}_z = \vec{e}_x$, and put $\vec{\beta} = \beta\vec{e}_x$. Then

$$G = \kappa^2(\vec{e}_\kappa \cdot \vec{e}_x)^2[\beta + p(\zeta)]f'f^{-2} \tag{3.9}$$

so that a necessary condition for stability when $g\rho_0' \equiv 0$
is

$$f'(\zeta)[\beta + p(\zeta)] \geq 0 \quad \text{on } [0,1] \tag{3.10}$$

The function $\beta + p(\zeta)$ is strictly increasing on $[0,1]$;
thus a value of β will exist such that equation (3.10)
holds if and only if for every pair $\zeta_1, \zeta_2 \in [0,1]$ for
which $f'(\zeta_1) < 0$ and $f'(\zeta_2) > 0$ we have $\zeta_1 < \zeta_2$. Such
a function $f(\zeta)$ must attain its maximum $(=1)$ at $\zeta = 0$
or $\zeta = 1$, and if $f(\zeta_0) = \min_{[0,1]} f(\zeta)$ (>0), then $f'(\zeta) \leq 0$
on $[0, \zeta_0]$ and $f'(\zeta) \geq 0$ on $[\zeta_0, 1]$, and equation (3.10)
will hold for $\beta = -p(\zeta_0)$. If $f' \geq 0$ on $[0,1]$, then any
positive β will do, while if $f' \leq 0$ on $[0,1]$, then any
$\beta < -p(1)$ will suffice. We conclude that when $g\rho_0' \equiv 0$,
the system will be exponentially unstable ($G < 0$ for all
$\kappa > 0$) for any $\eta_0(z)$ such that $\eta_0'(z_1) > 0$ and $\eta_0'(z_2) < 0$
where $0 \leq z_1 < z_2 \leq a$, and this holds for perfectly
conducting or insulating boundaries.

The function $Q_\zeta(\vec{e}_\kappa)$ is a bounded continuous
function of ζ and $\vec{e}_\kappa$ for any given positive $f \in C^2[0,1]$.
Therefore given any positive $\eta_0(z) \in C^2[0,a]$ and any

$\overline{B}_0^>(0)$, one can always make $G > 0$ on $[0,1]$ for arbitrary $\overline{k}^> \neq 0$ by taking $g\rho_0'$ sufficiently negative on $[0,1]$ (see equation (3.4)). This raises the possibility of "gravitationally" stabilizing the sheet pinch. Also, we have just seen that for certain functions $f(\zeta)$ we can make $G \geq 0$ even in the absence of gravity by choosing $\overline{\beta}^>$ $(\overline{B}_0^>(0))$ appropriately, which raises the possibility of magnetic stabilization. However, according to Theorem 3.1, $G \geq 0$ is only necessary for stability; the functionals F_ε must also be nonnegative if the system is to be stable. We now assume $G \geq 0$ and investigate the functionals F_ε. We assume also that η_0 is nonconstant and that $J^* > 0$, thereby excluding the trivial cases of $\overline{J}_0^> \equiv 0$ and constant η_0, which have been dealt with previously.

Case I. Insulating Boundaries

Set $\varepsilon = \kappa^2 > 0$, $\psi_0 = 1 + \kappa\zeta(1-\zeta) \in \widetilde{D}_5$, and let

$$n(\zeta) \equiv \frac{\kappa^2 r^2}{G + \kappa^2} = \frac{r^2}{(\overline{e}_\kappa^> \cdot \overline{b}^>)r - \gamma h' + 1} \geq 0 \qquad (3.11)$$

Note that n is independent of κ, and that for $\overline{\kappa}^>$ nonparallel to $\overline{e}_j^>$, we have $\int_0^1 n(\zeta)\,d\zeta > 0$. Then

$$F_{\kappa^2}[\psi_0] = \kappa^2 \int_0^1 \{(1-2\zeta)^2 + \psi_0^2\}\,d\zeta + 2\kappa - \int_0^1 n\psi_0^2\,d\zeta \qquad (3.12)$$

so that

$$\lim_{\kappa \to 0+} F_{\kappa^2}[\psi_0] = -\int_0^1 n \, d\zeta < 0 \qquad (\vec{\kappa} \times \vec{e}_j \neq 0) \qquad (3.13)$$

Thus for any n, however small, the sheet pinch will be exponentially unstable to disturbances with sufficiently small wavenumber k, provided $\vec{k}$ is not parallel to $\vec{J}_0$, η_0 is nonconstant, and $|\vec{J}_0| > 0$. Gravitational or magnetic stabilization is therefore not possible for such equilibria with insulating boundaries, unless some mechanism is introduced which bounds positive k away from zero (e.g., finite boundaries in the x and y directions). It is clear from equation (3.1) that the pinch will be exponentially stable to disturbances with wavenumber $\vec{k}$ satisfying

$$\kappa^2 \geq \sup_\zeta \left\{ \frac{r^2}{(\vec{e}_\kappa \cdot \vec{b}) r - \gamma h'} \right\}.$$

Case II. Perfectly Conducting Boundaries.

Every real $\psi(\zeta) \in \tilde{D}_5'$ satisfies $\int_0^1 (\psi')^2 \, d\zeta$ $\geq \pi^2 \int_0^1 \psi^2 \, d\zeta$, so that equation (3.1) gives

$$F_\epsilon[\psi] \geq \int_0^1 \left\{ \pi^2 + \kappa^2 - \frac{r^2}{(\vec{e}_\kappa \cdot \vec{b}) r - \gamma h' + \kappa^{-2} \epsilon} \right\} \psi^2 \, d\zeta$$

Therefore the system will be exponentially stable to disturbances with wave number $\vec{k}$ provided

$$\sup_\zeta \left\{ \frac{r^2}{(\vec{e}_\kappa \cdot \vec{b}) r - \gamma h'} \right\} \leq \pi^2 + \kappa^2 \qquad (3.14)$$

and will be stable for arbitrary disturbances if

$$\sup_{\kappa,\zeta} \left\{ \frac{r^2}{(\vec{e}_\kappa \cdot \vec{b})r - \gamma h'} \right\} \le \pi^2 \tag{3.15}$$

Gravitational stabilization is clearly always possible;
given any η_0 and $\vec{B}_0(0)$, equation (3.15) can be satisfied
by taking $g\rho_0'$ sufficiently large and negative. Magnetic
stabilization is possible for suitable (e.g., monotone) η_0;
sufficient conditions for exponential stability are that
$g\rho_0' \le 0$ on $[0,a]$, $\vec{B}_0(z)$ be unidirectional, equation (3.10)
holds, and

$$S(\zeta) \le \pi^2 , \qquad 0 \le \zeta \le 1, \tag{3.16}$$

where

$$S(\zeta) \equiv \begin{cases} f^{-2} \left| \dfrac{f'(\zeta)}{\beta + p(\zeta)} \right| , & \beta + p(\zeta) \ne 0 \\[2em] \lim_{\zeta \to \zeta_0} f^{-2} \left| \dfrac{f'(\zeta)}{\beta + p(\zeta)} \right| , & p(\zeta_0) = -\beta \end{cases} \tag{3.17}$$

for $0 \le \zeta \le 1$.

4. Stability Theorems

In this section we prove the assertion made previously: The pinch is exponentially stable to disturbances of wavenumber $\vec{k} = \vec{\kappa}\,a^{-1}$ if and only if $H(\vec{\kappa}) \geq 0$. This is accomplished by a straightforward application of several theorems, which we state and prove in an abstract setting; they are therefore by no means restricted in application to the specific problem of the MHD sheet pinch.

We consider functions $f(t)$, defined for $t \geq 0$, whose range is contained in some inner-product space with norm and inner product denoted by $\|\ \|$ and $(\ ,\ )$. Such a function $f(t)$ is said to be exponentially stable if for every $\varepsilon > 0$ there exists a constant $M_\varepsilon < \infty$ such that $\|f(t)\| \leq M_\varepsilon e^{\varepsilon t}$ for $t \geq 0$ (if $f(t)$ is real or complex-valued, the appropriate norm is of course the modulus of f). The two theorems to follow provide the sufficient conditions for exponential stability.

Theorem 4.1. Let P, H and iA be symmetric operators defined in an innner-product space, and suppose that $H \geq 0$ and $\delta \equiv \inf \dfrac{(\zeta, P\zeta)}{(\zeta, \zeta)} > 0$. Then if $f(t)$ is exponentially stable, so is every solution $\chi(t)$ of the equation

$$P\dot{\chi} + [H + A]\chi = f(t) \qquad t \geq 0 \qquad (4.1)$$

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Proof: $\frac{d}{dt}(\chi, P\chi) = (P\dot\chi, \chi) + (\chi, P\dot\chi)$

$$= (f - [H + A]\chi, \chi) + (\chi, f - [H + A]\chi)$$

$$= 2\ \text{Re}\ (f, \chi) - 2(\chi, H\chi) \leq 2\ \text{Re}\ (f, \chi).$$

Integrating from 0 to t and using the exponential stability of f(t) and the Schwarz inequality, we obtain

$$\delta \|\chi\|^2 \leq (\chi, P\chi) \leq (\chi_0, P\chi_0) + 2M_\varepsilon \int_0^t e^{\varepsilon\tau} \|\chi(\tau)\|\ d\tau$$

so that

$$\|\chi(t)\| \leq g(t) \equiv \left[\alpha + \beta \int_0^t e^{\varepsilon\tau} \|\chi\|\ d\tau\right]^{1/2} \tag{4.2}$$

where $\alpha \equiv \delta^{-1}(\chi_0, P\chi_0)$ and $\beta \equiv 2\delta^{-1} M_\varepsilon$.

Thus $\|\chi\|/g \leq 1$, and since $\dot g(t) = (\beta/2)g^{-1}\|\chi\|e^{\varepsilon t}$,

$$g(t) - g(0) = \left(\frac{\beta}{2}\right) \int_0^t e^{\varepsilon u} \|\chi\| g^{-1}(u)\ du \leq \left(\frac{\beta}{2}\right) \int_0^t e^{\varepsilon u}\ du$$

$$= \frac{\beta}{2\varepsilon} [e^{\varepsilon t} - 1].$$

From equation (4.2) we obtain

$$\|\chi(t)\| \leq g(0) + \beta(2\varepsilon)^{-1} e^{\varepsilon t} \leq [g(0) + \beta(2\varepsilon)^{-1}] e^{\varepsilon t}.$$

Theorem 4.2. Let P, K and H be nonnegative symmetric operators defined in an inner-product space, and let $\zeta(t)$ be any solution of the equation

$$P\ddot\zeta + K\dot\zeta + H\zeta = r(t), \qquad t \geq 0. \tag{4.3}$$

(A)   If $\|r(t)\| \leq M$, $\|\dot{r}(t)\| \leq N$ for $t \geq 0$, where M and N are constants, and if $\delta \equiv \inf \frac{(\xi, H\xi)}{(\xi, \xi)} > 0$, then there exist positive constants A, B, C, D and E such that

$$\|\zeta(t)\| \leq At + B , \qquad\qquad t \geq 0, \qquad\qquad (4.4)$$

$$(\dot{\zeta}, P\dot{\zeta}) \leq Ct^2 + Dt + E, \qquad\qquad t \geq 0. \qquad\qquad (4.5)$$

(B)   If $r(t)$ and $\dot{r}(t)$ are exponentially stable and if $\inf \frac{(\xi, [\alpha^2 P + \alpha K + H]\xi)}{(\xi, \xi)} > 0$ for all $\alpha > 0$, then ζ and $(\dot{\zeta}, P\dot{\zeta})$ are exponentially stable.

<u>Proof</u>:   (A) $\frac{d}{dt} \{(\dot{\zeta}, P\dot{\zeta}) + (\zeta, H\zeta)\} = (P\ddot{\zeta} + H\zeta, \dot{\zeta}) + (\dot{\zeta}, P\ddot{\zeta} + H\zeta)$

$$= (r - K\dot{\zeta}, \dot{\zeta}) + (\dot{\zeta}, r - K\dot{\zeta}) = 2 \, \text{Re}(r, \dot{\zeta}) - 2(\dot{\zeta}, K\dot{\zeta})$$

$$\leq 2 \, \text{Re} \, (r, \dot{\zeta}) = \frac{d}{dt} 2 \, \text{Re} \, (r, \zeta) - 2 \, \text{Re} \, (\dot{r}, \zeta) .$$

Integrating from 0 to t and using the Schwarz inequality, we obtain

$$(\dot{\zeta}, P\dot{\zeta}) + (\zeta, H\zeta) \leq \alpha + 2\|r\|\|\zeta\| + 2 \int_0^t \|\dot{r}\| \, \|\zeta\| \, du \qquad\qquad (4.6)$$

where $\alpha \equiv (\dot{\zeta}_0, P\dot{\zeta}_0) + (\zeta_0, H\zeta_0) - 2 \, \text{Re} \, (r_0, \zeta_0)$. Hence

$$\delta\|\zeta\|^2 \leq \alpha + 2M\|\zeta\| + 2N \int_0^t \|\zeta(u)\| \, du ,$$

so that

$$(\|\zeta\| - M\delta^{-1})^2 \leq \alpha\delta^{-1} + M^2\delta^{-2} + 2N\delta^{-1} \int_0^t \|\zeta\| \, du .$$

Set $p(t) \equiv \int_0^t \|\zeta\| \, du$, $A_1 = \alpha\delta^{-1} + M^2\delta^{-2}$, $B_1 \equiv 2N\delta^{-1}$.
Taking the square root of both sides of the above inequality, we obtain

$$\|\zeta(t)\| = \dot{p}(t) \leq (A_1 + B_1 p)^{1/2} + M\delta^{-1} \qquad (4.7)$$

so that

$$\frac{d}{dt} \{2B_1^{-1}(A_1 + B_1 p)^{1/2}\} = \dot{p}(A_1 + B_1 p)^{-1/2} \leq 1 + M\delta^{-1}A_1^{-1/2}.$$

Integrating from 0 to t yields

$$(A_1 + B_1 p)^{1/2} \leq (\frac{B_1}{2})(1 + M\delta^{-1}A_1^{-1/2})t + A_1^{1/2},$$

and equation (4.4) follows at once from equation (4.7). From equations (4.4) and (4.6) we find

$$(\dot{\zeta}, P\dot{\zeta}) \leq \alpha + 2M(At + B) + 2N \int_0^t (Au + B) \, du$$

which implies equation (4.5).

(B)   Let $\varepsilon > 0$, and set $\zeta(t) = e^{\varepsilon t}\xi(t)$. Then $\xi(t)$ satisfies

$$P\ddot{\xi} + K_\varepsilon \dot{\xi} + H_\varepsilon \xi = f(t) \equiv r\,e^{-\varepsilon t}, \quad t \geq 0,$$

where $K_\varepsilon \equiv 2\varepsilon P + K \geq 0$, $H_\varepsilon \equiv \varepsilon^2 P + \varepsilon K + H \geq 0$, and inf $(\xi, H_\varepsilon \xi)/(\xi, \xi) > 0$. Now $\|f\| = \|r\|e^{-\varepsilon t} \leq M_\varepsilon e^{\varepsilon t} e^{-\varepsilon t} = M_\varepsilon$ and $\|\dot{f}\| = \|\dot{r} - \varepsilon r\|e^{-\varepsilon t} \leq (\|\dot{r}\| + \varepsilon\|r\|)e^{-\varepsilon t} \leq N_\varepsilon + \varepsilon M_\varepsilon$ so

that by (A), $\|\xi(t)\| \le At + B$ for $t \ge 0$. Therefore
$\|\zeta(t)\| = \|\xi\| e^{\varepsilon t} \le (At + B)e^{\varepsilon t}$ for $t \ge 0$, which implies
that ζ is exponentially stable. Since r, $\dot{r}$, and ζ are
exponentially stable, given any $\varepsilon > 0$ there exist
constants M_ε, N_ε and $\tilde{M}_\varepsilon$ such that $\|r\| \le M_\varepsilon e^{\varepsilon t}$,
$\|\dot{r}\| \le N_\varepsilon e^{\varepsilon t}$, and $\|\zeta\| \le \tilde{M}_\varepsilon e^{\varepsilon t}$ for $t \ge 0$. It follows
from equation (4.6) that

$$(\dot{\zeta}, P\dot{\zeta}) \le \alpha + 2M_\varepsilon \tilde{M}_\varepsilon e^{2\varepsilon t} + 2N_\varepsilon \tilde{M}_\varepsilon \int_0^t e^{2\varepsilon u}\, du$$

$$\le [\alpha + 2M_\varepsilon \tilde{M}_\varepsilon + \varepsilon^{-1} N_\varepsilon \tilde{M}_\varepsilon] e^{2\varepsilon t}$$

which proves the exponential stability of $(\dot{\zeta}, P\dot{\zeta})$.

Exponential stability of the pinch for $H \ge 0$
follows easily from Theorems 4.1 and 4.2. The crucial
equations are (2.25), (2.34), and (2.81) (or its
equivalent (2.46)). These are differential equations
in t for the quantities R, J_z, W (or ξ), and ψ (or B_z);
the remaining variables are all obtained algebraically
from this fundamental set and its derivatives with
respect to t and z. We show that $H \ge 0$ implies the
exponential stability of R, J_z, W and ψ in the L_2 norm.
Suppose then that $H \ge 0$, where H is given by equation
(2.84). It is a simple matter to verify that P, K, and
H, given by equations (2.82)-(2.84) with domains
(2.94)-(2.96), satisfy the hypothesis of Theorem 4.2 (B);

C, given by equation (2.90), is independent of t so that
C and $\dot{C}$ are exponentially stable; therefore Theorem 4.2 (B)
implies that the solution $\Theta(t)$ of equation (2.81) is
exponentially stable, as is $(\dot{\Theta}, P\dot{\Theta})$. This immediately
establishes the exponential stability of W, ψ, $\dot{W}$, $\dot{W}'$,
and therefore of ξ, B_z, $\dot{\xi}$, and $\dot{\xi}'$. From the fact that
$\frac{d}{dt} \|\xi'\| \le \|\dot{\xi}'\|$ for $\|\xi'\| > 0$, we infer that ξ' is
exponentially stable. Equations (2.25) and (2.34) can
be written as the matrix system

$$P\dot{\chi} + (H + A)\chi = f$$

with

$$P = \begin{pmatrix} \rho_0 & 0 \\ 0 & \mu_0 \end{pmatrix}, \quad H = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -i\mu_0 F \\ -i\mu_0 F & 0 \end{pmatrix}$$

$$\chi = \begin{pmatrix} R \\ J_z \end{pmatrix}, \quad f = \begin{pmatrix} qB_z \\ -q\mu_0\dot{\xi} - \frac{d}{dz} \{q[\eta_0'\xi - \eta(z,0) - \eta_0'\xi_0]\} \end{pmatrix}$$

The hypothesis of Theorem 4.1 is satisfied, and we
conclude that R and J_z are exponentially stable.

The case $\vec{k} = 0$ requires separate consideration.
In this circumstance, the perturbed quantities are
functions of z and t only, and equations (2.17)-(2.21)
and (2.23) and boundary conditions (2.50)-(2.51) lead
to (we omit the subscript 1 on the perturbed variables)

$$V_z \equiv 0, \quad J_z \equiv 0, \quad B_z = \text{constant}$$

$$\rho(z,t) = \rho(z,0), \qquad \eta(z,t) = \eta(z,0)$$

$$P(z,t) = -g \int_0^z \rho(u,0) \, du - \mu_0^{-1} \vec{B} \cdot \vec{B}_0 + \text{constant}$$

$$\rho_0 \dot{V}_x + L_1 V_x = \eta_0^{-1} E_{0y} B_z, \quad V_x(0,t) = V_x(a,t) = 0$$
$$(\nu_0 > 0)$$

$$\rho_0 \dot{V}_y + L_1 V_y = -\eta_0^{-1} E_{0x} B_z, \quad V_y(0,t) = V_y(a,t) = 0$$
$$(\nu_0 > 0)$$

$$\mu_0 J_x = - B_y', \qquad \mu_0 J_y = B_x',$$

$$\mu_0 \dot{B}_x + L_2 B_x = (\eta B_{0x}')', \quad B_x(0,t) = B_x(a,t) = 0$$

$$\mu_0 \dot{B}_y + L_2 B_y = (\eta B_{0y}')', \quad B_y(0,t) = B_y(a,t) = 0 \, .$$

The boundary conditions imposed on B_x and B_y correspond to perturbations with zero net electric current, i.e., they imply and are implied by perturbations satisfying $\int_0^a \vec{J}(z,t) \, dz = 0$. The operators L_1 and L_2 are nonnegative for the given boundary data, and exponential stability follows from Theorem 4.1.

Having established the sufficiency of $H \geq 0$ for exponential stability, we turn to the demonstration of its necessity. We give two proofs. The first is quite general in application but assumes the existence of solutions to equation (2.81) satisfying certain initial data -- an assumption obviously justified (indeed required) on physical grounds and reasonable mathematically, as

the initial data and the equilibrium quantities can be chosen to be C^∞ functions. The second proof makes no assumptions -- there we show that if H is not positive semidefinite, equation (2.81) with $C \equiv 0$ possesses a normal-mode solution $\Theta(z,t) = \Theta(z)e^{\omega t}$ with $\omega > 0$. Note that it is sufficient to demonstrate the existence of an exponentially growing solution to the homogeneous equation (2.81), since given any $\xi_0(z)$, we may choose $\eta(z,0)$ and $\rho(z,0)$ such that $C \equiv 0$ (see equation (2.46)).

Theorem 4.3. Let P, K, and H be symmetric operators defined in an inner-product space, $P \geq 0$, $K \geq 0$, and $-\infty < \delta \equiv \inf H < 0$. Suppose that for all sufficiently small positive ε the equation $P\ddot{\zeta} + K\dot{\zeta} + H\zeta = 0$ admits of a solution $\zeta(t;\varepsilon)$ for $t \geq 0$ satisfying the initial data $\zeta(0;\varepsilon) = \eta$, $\dot{\zeta}(0;\varepsilon) - \varepsilon\eta \in N_P$, where N_P is the nullspace of P, $\zeta \in D_P \cap D_K \cap D_H$ for $t \geq 0$, and $(\eta,H\eta) < 0$. Then there exists positive constants α and β such that $\|\zeta(t;\alpha)\| \geq \beta \, e^{\alpha t}$ for $t \geq 0$.

Proof: Let $\varepsilon \geq 0$, and define $H_\varepsilon \equiv \varepsilon^2 P + \varepsilon K + H$ and $K_\varepsilon \equiv 2\varepsilon P + K$. Then $(\eta,H_0\eta) = (\eta,H\eta) < 0$, and since $(\eta,H_\varepsilon\eta)$ is a continuous function of ε, there exists $\alpha > 0$ such that $(\eta,H_\alpha\eta) < 0$ and $\zeta(t;\alpha)$ exists. Let $\xi(t) \equiv \zeta(t;\alpha)e^{-\alpha t}$. Then $\xi(t)$ satisfies the equation $P\ddot{\xi} + K_\alpha\dot{\xi} + H_\alpha\xi = 0$, and we have $\xi_0 = \zeta(0;\alpha) = \eta$,

$\dot{\xi}_0 = \dot{\zeta}(0;\alpha) - \alpha\zeta(0;\alpha) = \dot{\zeta}(0;\alpha) - \alpha\eta \in N_P$. Now

$\frac{d}{dt}\{(\dot{\xi},P\dot{\xi}) + (\xi,H_\alpha\xi)\} = -2(\dot{\xi},K_\alpha\dot{\xi}) \leq 0$, so that

$\delta\|\xi\|^2 \leq (\xi,H_\alpha\xi) \leq (\dot{\xi},P\dot{\xi}) + (\xi,H_\alpha\xi) \leq (\dot{\xi}_0,P\dot{\xi}_0) + (\xi_0,H\xi_0)$

$= (\eta,H_\alpha\eta) < 0$, and we have $\|\xi(t)\|^2 \geq \delta^{-1}(\eta,H_\alpha\eta) > 0$

for $t \geq 0$. Therefore $\|\zeta(t;\alpha)\| = \|\xi\|e^{\alpha t} \geq [\delta^{-1}(\eta,H_\alpha\eta)]^{1/2}e^{\alpha t}$.

<u>Theorem</u> 4.4. Suppose $\nu_0 \equiv 0$, $\kappa > 0$, and that $f(\zeta)$ and $h(\zeta)$ are positive continuously differentiable functions on $[0,1]$. Let $\inf\limits_{D_H} \frac{\langle\Theta,H\Theta\rangle}{\langle\Theta,\Theta\rangle} < 0$. Then there exists a nontrivial solution to the equation

$$P\ddot{\Theta} + K\dot{\Theta} + H\Theta(\zeta,t) = 0 \tag{4.8}$$

(the homogeneous form of equation (2.81)) of the form

$\Theta(\zeta,t) = \begin{pmatrix} W(\zeta) \\ \psi(\zeta) \end{pmatrix} e^{\Omega t}$ where Ω is a positive constant.

<u>Proof</u>. Let $\mathcal{H}$ denote the Hilbert space $L_2[0,1]$, $\mathcal{H}^2 \equiv L_2[0,1] \times L_2[0,1]$, and let D_5 denote either $\tilde{D}_5$ or $\tilde{D}_5'$. The hypotheses on h and κ guarantee the existence of the positive compact Hermitian integral operators K_3 and K_5 on $\mathcal{H}$ possessing the following properties:

$$K_3\tilde{L}_3 = I \quad \text{on } \tilde{D}_3 , \qquad K_5\tilde{L}_5 = I \quad \text{on } D_5 \tag{4.9}$$

$$\tilde{L}_3K_3 = I \quad \text{on } C[0,1], \quad \tilde{L}_5K_5 = I \quad \text{on } C[0,1] \tag{4.10}$$

$$K_3(\mathcal{H}) \subset C[0,1], \qquad K_5(\mathcal{H}) \subset C[0,1] \tag{4.11}$$

Define $k_3 \equiv K_3^{1/2}$, $k_5 \equiv K_5^{1/2}$. Then k_3 and k_5 are positive compact Hermitian operators on $\mathcal{H}$. It will be convenient to denote the elements of the symmetric matrix operators K and H by K_{ij} and H_{ij} $(i,j=1,2)$. For each real ω we define the operators

$$P_\omega \equiv \begin{pmatrix} \omega^2 I & 0 \\ 0 & I \end{pmatrix}, \quad -C_\omega \equiv \begin{pmatrix} k_3[\omega K_{11}+H_{11}]k_3 & k_3[\omega K_{12}+H_{12}]k_5 \\ k_5[\omega K_{21}+H_{21}]k_3 & k_5\omega K_{22}k_5 \end{pmatrix}$$

The hypothesis on f and h insures that H_{11}, H_{12}, H_{21} as well as all the K_{ij} are continuous functions of ζ on $[0,1]$. Thus C_ω is compact. P_ω and C_ω are clearly Hermitian. Let $F(\omega) \equiv \inf_{\mathcal{H}^2} \dfrac{\langle \xi, [P_\omega - C_\omega]\xi \rangle}{\langle \xi, \xi \rangle}$. $F(\omega)$ is a real-valued continuous function of ω on $(-\infty, \infty)$. It is not difficult to show that $F(\omega)$ is nonnegative for all sufficiently large ω, and that $\inf_{D_H} \dfrac{\langle \theta, H\theta \rangle}{\langle \theta, \theta \rangle} < 0$ implies $F(0) < 0$. Therefore there exists $\Omega > 0$ such that $F(\Omega) = 0$, and the definition of $F(\Omega)$ yields $\sup_{\mathcal{H}^2} \dfrac{\langle \xi, C_\Omega \xi \rangle}{\langle \xi, P_\Omega \xi \rangle} = 1$. It follows from well-known theorems on compact Hermitian operators that there exists a nontrivial $\xi = \begin{pmatrix} \xi_1(\zeta) \\ \xi_2(\zeta) \end{pmatrix} \in \mathcal{H}^2$ such that $P_\Omega \xi = C_\Omega \xi$. Let $W(\zeta) \equiv k_3 \xi_1$ and $\psi(\zeta) \equiv k_5 \xi_2$. Then we have

$$\Omega^2 W = -K_3\{[\Omega K_{11} + H_{11}]W + [\Omega K_{12} + H_{12}]\psi\} \tag{4.12}$$

$$\psi = -K_5\{[\Omega K_{21} + H_{21}]W + [\Omega K_{22}]\psi\} \tag{4.13}$$

It follows at once from (4.11) that W and ψ are continuous on [0,1]; (4.10) then implies that $W \in \tilde{D}_3$ and $\psi \in D_5$. Multiplying both sides of equations (4.12) and (4.13) on the left with $\tilde{L}_3$ and $\tilde{L}_5$, respectively, we obtain

$$\{\Omega^2 P + \Omega K + H\} \begin{pmatrix} W \\ \psi \end{pmatrix} = 0 \; ,$$

and the proof is complete.

Acknowledgements

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N.Y.U. Courant Institute of
Mathematical Sciences
251 Mercer St.
New York, N. Y. 10012